

Cubic B-Splines for Nonequally Spaced Nodes

As explained in Section 5.4.3, the cubic spline interpolation function can be written as

$$s(x) = \sum_{i=0}^{n+2} a_i B_i(x), \quad (1)$$

where $B_i(x)$ is the cubic B-spline with center node x_i . The data nodes are assumed to be given and satisfy $x_1 < x_2 < \dots < x_{n+1}$. The sum in (1) includes every cubic B-spline that is nonzero somewhere on $x_1 \leq x \leq x_{n+1}$. Moreover, by the way it is constructed, $s \in C^2(-\infty, \infty)$.

In what follows, let $h_i = x_{i+1} - x_i$ and

$$(z)_+ = \max(z, 0).$$

Also, let

$$\begin{aligned} H_i &= h_{i-1} + h_i = x_{i+1} - x_{i-1}, \\ K_i &= h_{i-1} + h_i + h_{i+1} = x_{i+2} - x_{i-1}, \\ L_i &= x_{i+2} - x_{i-2}, \end{aligned}$$

and

$$\begin{aligned} A_i^L &= \frac{1}{h_{i-2}H_{i-1}K_{i-1}}, & A_i^R &= \frac{1}{h_{i+1}H_{i+1}K_i}, \\ C_i^L &= \frac{L_i}{h_{i-2}h_{i-1}H_iK_i}, & C_i^R &= \frac{L_i}{h_ih_{i+1}H_iK_{i-1}}. \end{aligned}$$

The cubic B-spline with support $[x_{i-2}, x_{i+2}]$ is then

$$B_i(x) = \begin{cases} A_i^L(x - x_{i-2})^3 - C_i^L(x - x_{i-1})_+^3, & x_{i-2} \leq x \leq x_i, \\ A_i^R(x_{i+2} - x)^3 - C_i^R(x_{i+1} - x)_+^3, & x_i \leq x \leq x_{i+2}, \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

As required, on each of the four subintervals B_i is a cubic, and B_i , B'_i , and B''_i are continuous at all the nodes. The nodes outside the interval $x_1 \leq x \leq x_{n+1}$ will be specified when discussing the end conditions.

As defined above, the B-splines are nonnegative and form a partition of unity, which in this case means that

$$\sum_{k=0}^{n+2} B_k(x) = 1 \quad \text{for } x_1 \leq x \leq x_{n+1}. \quad (3)$$

Values at the nodes

The values of $B_i(x)$ and its first two derivatives at the nodes are given in Table 1. This generalizes Table 5.3 in the text.

	x_{i-1}	x_i	x_{i+1}	Other nodes
B_i	$A_i^L h_{i-2}^3$	$A_i^L H_{i-1}^3 - C_i^L h_{i-1}^3$	$A_i^R h_{i+1}^3$	0
B_i'	$3A_i^L h_{i-2}^2$	$3(A_i^L H_{i-1}^2 - C_i^L h_{i-1}^2)$	$-3A_i^R h_{i+1}^2$	0
B_i''	$6A_i^L h_{i-2}$	$6(A_i^L H_{i-1} - C_i^L h_{i-1})$	$6A_i^R h_{i+1}$	0

Table 1: Values of the cubic B-spline $B_i(x)$ given in (2) and its first two derivatives at the nodes.

For nonequally spaced nodes $B_i(x)$ is generally not symmetric about x_i . So, for example, $B_i'(x_i)$ need not be zero. It is symmetric about x_i if $h_{i-2} = h_{i+1}$ and $h_{i-1} = h_i$, which does not require all four spacings to be equal.

Finding the coefficients

At x_i , only B_{i-1} , B_i , and B_{i+1} are nonzero. Define

$$\lambda_i^L = B_{i-1}(x_i) = A_{i-1}^R h_i^3 = \frac{h_i^2}{H_i K_{i-1}}, \quad (4)$$

$$\lambda_i^R = B_{i+1}(x_i) = A_{i+1}^L h_{i-1}^3 = \frac{h_{i-1}^2}{H_i K_i}, \quad (5)$$

$$\lambda_i = B_i(x_i) = \frac{h_i H_{i-1}}{H_i K_{i-1}} + \frac{h_{i-1} H_{i+1}}{H_i K_i}. \quad (6)$$

The partition of unity in (3) gives

$$\lambda_i^L + \lambda_i + \lambda_i^R = 1.$$

Thus $\lambda_i = 1 - \lambda_i^L - \lambda_i^R$, but the positive sum in (6) is better for computing this coefficient when the spacings vary greatly and λ_i is small.

The interpolation conditions $s(x_i) = y_i$ become

$$\lambda_i^L a_{i-1} + \lambda_i a_i + \lambda_i^R a_{i+1} = y_i, \quad \text{for } i = 1, \dots, n+1. \quad (7)$$

These have the three-term structure obtained for equally spaced nodes. To complete the formulation the two end conditions must be specified.

Endpoint construction for a natural spline

Suppose the data are (x_i, y_i) , $i = 1, 2, \dots, n+1$, with $n \geq 1$. The natural end conditions are

$$s''(x_1) = s''(x_{n+1}) = 0. \quad (8)$$

One simple choice is to extend the nodes using the spacing of the first and last data intervals:

$$x_{1-j} = x_1 - jh_1, \quad x_{n+1+j} = x_{n+1} + jh_n, \quad j = 1, 2, 3. \quad (9)$$

With this, and using a reduction similar to the one given on page 223 of the text, one finds that

$$a_1 = y_1, \quad a_{n+1} = y_{n+1}.$$

Consequently, the complete tridiagonal system for $a_1, \dots, a_{n+1}$ is

$$\begin{aligned} a_1 &= y_1, \\ \lambda_i^L a_{i-1} + \lambda_i a_i + \lambda_i^R a_{i+1} &= y_i, \quad i = 2, \dots, n, \\ a_{n+1} &= y_{n+1}. \end{aligned} \tag{10}$$

The two exterior coefficients are then recovered from

$$\begin{aligned} a_0 &= a_1 + \frac{3h_1}{K_1}(a_1 - a_2), \\ a_{n+2} &= a_{n+1} + \frac{3h_n}{K_n}(a_{n+1} - a_n). \end{aligned} \tag{11}$$

Endpoint construction for a clamped spline

Suppose $n \geq 1$, use the exterior nodes in (9), and prescribe the endpoint slopes

$$s'(x_1) = d_L, \quad s'(x_{n+1}) = d_R. \tag{12}$$

Table 1 gives

$$\begin{aligned} d_L &= \frac{a_1 - a_0}{2h_1} + \frac{3(a_2 - a_1)}{2K_1}, \\ d_R &= \frac{a_{n+2} - a_{n+1}}{2h_n} + \frac{3(a_{n+1} - a_n)}{2K_n}. \end{aligned} \tag{13}$$

Thus the exterior coefficients can be eliminated using

$$\begin{aligned} a_0 &= a_1 + \frac{3h_1}{K_1}(a_2 - a_1) - 2h_1 d_L, \\ a_{n+2} &= a_{n+1} + \frac{3h_n}{K_n}(a_n - a_{n+1}) + 2h_n d_R. \end{aligned} \tag{14}$$

Define

$$\rho_L = \frac{h_1}{K_1}, \quad \rho_R = \frac{h_n}{K_n}.$$

Substituting (14) into the first and last interpolation equations gives the complete tridiagonal system

$$\begin{aligned} (1 - \rho_L)a_1 + \rho_L a_2 &= y_1 + \frac{h_1 d_L}{3}, \\ \lambda_i^L a_{i-1} + \lambda_i a_i + \lambda_i^R a_{i+1} &= y_i, \quad i = 2, \dots, n, \\ \rho_R a_n + (1 - \rho_R)a_{n+1} &= y_{n+1} - \frac{h_n d_R}{3}. \end{aligned} \tag{15}$$

After solving for $a_1, \dots, a_{n+1}$, use (14) to recover a_0 and a_{n+2} . For two data points ($n = 1$), the middle equations are absent and the spline agrees on the data interval with the cubic determined by the two values and endpoint slopes.

Endpoint construction for a not-a-knot spline

Suppose $n \geq 3$, so there are at least four data points, and use the same exterior nodes as in (9). The not-a-knot conditions are

$$s'''(x_2^-) = s'''(x_2^+), \quad s'''(x_n^-) = s'''(x_n^+). \quad (16)$$

Since s , s' , and s'' are already continuous, these conditions make the first two cubic pieces coincide, and likewise the last two. Thus x_2 and x_n are not needed as knots of the resulting spline on the data interval.

For convenience, define

$$D_i = C_i^L + C_i^R - A_i^L - A_i^R = \frac{L_i}{h_{i-1}h_iH_{i-1}H_{i+1}}.$$

Differentiating (2) three times shows that the jumps in B_i''' , with each jump defined as the right-hand limit minus the left-hand limit, are

$$6A_i^L, \quad -6C_i^L, \quad 6D_i, \quad -6C_i^R, \quad 6A_i^R$$

at $x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}$, respectively. Consequently, after dividing by 6, the two not-a-knot equations are

$$A_{k-2}^R a_{k-2} - C_{k-1}^R a_{k-1} + D_k a_k - C_{k+1}^L a_{k+1} + A_{k+2}^L a_{k+2} = 0, \quad k = 2, n. \quad (17)$$

Although $B_{k-2}(x_k) = B_{k+2}(x_k) = 0$, their third derivatives have jumps at x_k , so five coefficients enter each of these equations.

The resulting system consists of (17) together with the interpolation equations (7). These $n+3$ equations uniquely determine the coefficients $a_0, \dots, a_{n+2}$. Placing the left not-a-knot equation first, the interpolation equations next in increasing node order, and the right not-a-knot equation last gives a banded system. The interpolation rows each contain three nonzero entries, and the two end rows each contain five. As shown below, it can also be reduced to a tridiagonal system.

Eliminating the exterior coefficients and using the endpoint interpolation equations gives $a_2 = \beta_L$ and $a_n = \beta_R$, where

$$\begin{aligned} \beta_L &= \frac{(h_1 + h_2)^3 y_2 - h_2^3 y_1 - h_1^3 y_3}{3h_1 h_2 (h_1 + h_2)}, \\ \beta_R &= \frac{(h_{n-1} + h_n)^3 y_n - h_n^3 y_{n-1} - h_{n-1}^3 y_{n+1}}{3h_{n-1} h_n (h_{n-1} + h_n)}. \end{aligned} \quad (18)$$

The reduced tridiagonal system is therefore

$$\begin{aligned} a_2 &= \beta_L, \\ \lambda_i^L a_{i-1} + \lambda_i a_i + \lambda_i^R a_{i+1} &= y_i, \quad i = 3, \dots, n-1, \\ a_n &= \beta_R. \end{aligned} \quad (19)$$

The remaining coefficients follow from the interpolation equations with $i = 2, n, 1, n+1$, in that order.

For numerical evaluation, β_L and β_R can be expressed in terms of differences between neighboring data values. For example,

$$\beta_L = y_2 + \frac{h_2^2(y_2 - y_1)}{3h_1(h_1 + h_2)} + \frac{h_1^2(y_2 - y_3)}{3h_2(h_1 + h_2)}.$$

Similar formulas for the second derivatives of the spline at the endpoints are also used. These avoid division by interpolation weights that may be very small.

data interval. If there are only three data points ($n = 2$), the two not-a-knot conditions coincide; the usual convention is to use the interpolating quadratic. With two data points ($n = 1$), use the line through the data on the data interval.

MATLAB implementation

The accompanying function uses the cubic B-spline basis above and the Thomas algorithm for each of the three tridiagonal systems. The calls are

```
y = spline3(xd,yd,x,'natural');
y = spline3(xd,yd,x,'not-a-knot');
y = spline3(xd,yd,x,'clamped',[dL,dR]);
```

Here `xd` and `yd` contain the data, `x` contains the evaluation points, and `dL` and `dR` are the endpoint derivative values. The default is `'not-a-knot'`.

Reduction to equally spaced nodes

If $h_j = h$, then $H_i = 2h$, $K_i = 3h$, and $L_i = 4h$, so the four coefficients become

$$A_i^L = A_i^R = \frac{1}{6h^3}, \quad C_i^L = C_i^R = \frac{4}{6h^3}.$$

Substituting these values into (2) gives (5.23) in the text, and Table 1 reduces to Table 5.3 in the text. Similarly, (7) reduces to (5.26). Finally, (17) reduces to the correct equation, which is

$$a_{k-2} - 4a_{k-1} + 6a_k - 4a_{k+1} + a_{k+2} = 0, \quad k = 2, n.$$

For natural end conditions, (11) becomes

$$a_0 = 2a_1 - a_2, \quad a_{n+2} = 2a_{n+1} - a_n.$$

For clamped end conditions, the first and last equations in (15) become

$$2a_1 + a_2 = 3y_1 + hd_L, \quad a_n + 2a_{n+1} = 3y_{n+1} - hd_R,$$

and (14) becomes

$$a_0 = a_2 - 2hd_L, \quad a_{n+2} = a_n + 2hd_R.$$

The reduced not-a-knot boundary values in (18) reduce to

$$\beta_L = \frac{8y_2 - y_1 - y_3}{6}, \quad \beta_R = \frac{8y_n - y_{n-1} - y_{n+1}}{6}.$$